Synchronization in a network of neuronal oscillators with finite storage capacity

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(Received 24 March 1995)

We investigate the synchronization phenomena in a network of neuronal oscillators with finite storage capacity. The effective Hamiltonian describing the stationary state of the system is analyzed via the replica method, to yield a phase diagram in the three-dimensional parameter space. The system is found to display a variety of behaviors including first-order and second-order transitions as well as reentrance.

PACS number(s): 87.10.+e, 02.50.−r, 05.70.Fh

I. INTRODUCTION

The oscillatory behaviors of neurons observed in real biological systems have been suggested to play important roles in various types of information processing, such as sensory segmentation, binding, and motor control [1–3]. In particular, the oscillatory activity in the primary visual cortex of a cat indeed appears to support the view that the oscillation could be crucial to the feature linking [4]. Here, one of the remarkable features of an assembly of oscillators is the emergence of a coherent motion over a relatively long distance, which is called *collective synchronization* [5]. The temporal synchronization in the cortex suggests that the processing of information is cooperative and involves neurons with different states. Through analytical and numerical studies of oscillatory networks, many insights have been obtained into this coherent behavior of diverse biological organisms [6]. In those studies, the individual neuron is usually modeled as a self-oscillatory functional unit under suitable conditions.

Recently, Arenas and Vincente studied a network of coupled neuronal oscillators and pointed out that phase locking can serve as a mechanism for memory storage in the system, which is destroyed if the distribution of natural frequencies is sufficiently broad [7]. In the study, the state of each neuron is described by the phase, leading to the well-known phase model. The resulting Arenas-Vincente (AV) model has similarity to the Q-state clock model of neural networks [8] in the limit $Q \to \infty$. The results of the two models agree with each other in the proper limit. However, both models are limited to special cases. The AV model is restricted to the low loading limit where only a finite number of patterns are stored; the Q-state model does not include the effects of the distribution of natural frequencies.

In this paper, we generalize the AV model to the case of finite storage capacity and obtain the phase diagram in the three-dimensional parameter space, which reproduces the results of the existing models in the appropriate limits. The system can serve as an attractor neural network, which stores information through mutual phase locking and exhibits interesting behaviors including first-order and second-order transitions as well as reentrance. When all the oscillators are identical, the network is found to exhibit behaviors qualitatively the same as those of the Q-state neural networks and the Hopfield model. Thus phase locking is destroyed when the storage exceeds the critical value. The appearance of a glassy phase through a continuous transition is also pointed out; here the distribution of natural frequencies makes the glass transition discontinuous.

The paper is organized as follows. In Sec. II we derive the effective Hamiltonian from the Fokker-Planck equation governing the time evolution of the system and use the replica method to obtain the self-consistency equations for the order parameters. The numerical study of the self-consistency equations is presented in Sec. III. The critical value of the storage capacity, beyond which phase locking does not persist, is found and the corresponding transition line separating the synchronized and the desynchronized phases is calculated. Interesting reentrant behavior is also revealed in the resulting phase diagram obtained in the three-dimensional parameter space. Further, the existence of a glassy phase is pointed out, which appears via a continuous or a discontinuous transition according to the distribution of natural frequencies. Finally, Sec. IV gives a brief discussion as well as a summary of the main result.

II. EFFECTIVE HAMILTONIAN AND EQUATIONS OF STATES

We follow Ref. [7] and begin with the Langevin equation governing the dynamics of the system

$$\frac{d\theta_i}{dt} = \omega_i - \sum_{j=1}^{N} J_{ij} \sin(\theta_i - \theta_j) + \gamma_i(t), \tag{1}$$

where θ_i and ω_i represent the phase and the natural frequency of the ith oscillator, respectively, J_{ij} is the coupling matrix, N is the size of the population, and $\gamma_i(t)$'s are independent white noises with zero mean and correlation

$$\langle \gamma_i(t)\gamma_j(t')\rangle = 2D\delta_{ij}\delta(t-t'), \quad D \geq 0. \tag{2}$$

Without loss of generality, we set the mean natural frequency equal to zero and assume that ω_i's are distributed according to a Gaussian distribution with the variance σ. The synaptic couplings contain information to be stored and are constructed as

$$J_{ij} = \frac{J}{N} \sum_{\mu=1}^{\alpha N} \cos(\xi_i^\mu - \xi_j^\mu), \quad \xi \in [0, 2\pi]. \tag{3}$$

Hereafter, we set $J \equiv 1$ for simplicity and make all the parameters dimensionless. Namely, the time t is measured in units of $1/J$.

Our goal is to determine the stationary properties of Eq. (1) for finite α. For this purpose, a convenient way is to introduce the appropriate probability density and consider the Fokker-Planck equation corresponding to the Langevin equation (1). Here the simple case of $\alpha = 0$, together with the help of order parameters measuring the correlations between the state of the system and the patterns to be stored, allows us to reduce Eq. (1) to the Fokker-Planck equation for the *one-oscillator* probability density. Unfortunately, however, this simple reduction does not work for finite α due to the lack of the self-averaging property. We thus resort to the Fokker-Planck equation for the *N-oscillator* probability density $P(\{\theta_i\}, \{\omega_i\}, \xi, t)$:

$$\frac{\partial P}{\partial t} = -\sum_i \left[\frac{\partial}{\partial \theta_i} h_i - D \frac{\partial^2}{\partial \theta_i^2} \right] P \ , \tag{4}$$

with

$$h_i = \omega_i - \sum_j J_{ij} \sin(\theta_i - \theta_j),$$

which leads to a stationary solution $P^{(0)}(\{\theta_i\}) \propto \exp(A[\theta])$ with action

$$A[\theta] = \frac{1}{D} \left[\sum_i \omega_i \theta_i + \sum_{i<j} J_{ij} \cos(\theta_i - \theta_j) \right] .$$

It is obvious that this solution has the form of a Gibbs measure with the *effective Hamiltonian*

$$H = -\frac{1}{2N} \sum_\mu \left\{ \left[\sum_i \cos(\theta_i - \xi_i^\mu) \right]^2 + \left[\sum_i \cos(\theta_i + \xi_i^\mu) \right]^2 + \left[\sum_i \sin(\theta_i - \xi_i^\mu) \right]^2 + \left[\sum_i \sin(\theta_i + \xi_i^\mu) \right]^2 \right\} - 2 \sum_i \omega_i \theta_i, \tag{5}$$

at temperature $T \equiv 2D$. In principle, the proper solution should be periodic in θ: $P(\theta) = P(\theta + 2\pi)$ or $A(\theta) = A(\theta + 2\pi)$, and the above solution does not appear to be adequate for describing the dynamics of the system governed by Eq. (1), which involves variations of θ larger than 2π. Similar problems arise in superconducting arrays driven by external currents [9], where the action was regarded as a periodic function with period $2n\pi$ ($n \to \infty$). Then it was pointed out that the standard Villain approximation, which gives an accurate description at low temperatures, restores the correct periodicity and yields results independent of n. We thus follow Ref. [9] and regard H in Eq. (5) as the effective Hamiltonian of the system, with the period $2n\pi$. Here, similarly to Ref. [9], it can be seen that the corresponding free energy is independent of n. Therefore we take the integration interval of θ to be from $-\pi$ to π. The free energy functional can then be obtained via the replica method in the thermodynamic limit $N \to \infty$.

The replica method allows us to write the partition function in the form of the multiple integral [10]

$$Z^n = \sum_{[\theta^a]} \left(\frac{\beta N}{2\pi} \right)^{2\alpha n N} \int \prod_{a=1}^{n} \prod_{\mu=1}^{\alpha N} dA_a^\mu d\widetilde{A}_a^\mu dB_a^\mu d\widetilde{B}_a^\mu \times \prod_{a,\mu} X_\mu \exp\left(2\beta \sum_i \omega_i \theta_i^a \right), \tag{6}$$

where $\beta \equiv T^{-1}$ is the inverse temperature and X_μ stands for

$$X_\mu = \exp\Bigg\{ -\frac{\beta}{2} \sum_{a,i} \Big[(A_a^\mu)^2 + (\widetilde{A}_a^\mu)^2 + (B_a^\mu)^2 + (\widetilde{B}_a^\mu)^2 - 2A_a^\mu \cos(\theta_i^a - \xi_i^\mu) - 2\widetilde{A}_a^\mu \cos(\theta_i^a + \xi_i^\mu) - 2B_a^\mu \sin(\theta_i^a - \xi_i^\mu) - 2\widetilde{B}_a^\mu \sin(\theta_i^a + \xi_i^\mu) \Big] \Bigg\}.$$

For simplicity, we consider the case that only the first pattern ($\mu = 1$) is condensed, while others have negligible overlap with the state of the system: $A_a^\mu, \widetilde{A}_a^\mu, B_a^\mu, \widetilde{B}_a^\mu \ll 1$ for $\mu \neq 1$. Hereafter, the index $\mu = 1$ will be omitted. After taking the average over noncondensed patterns [10], we obtain the free energy functional per oscillator

$$f = \lim_{n\to 0} \frac{1}{N} \left[1 - \left(\frac{\beta N}{2\pi} \right)^{2\pi} \int \prod_{\alpha=1}^{4n} dm_\alpha \prod_{\alpha<\beta} dq_{\alpha\beta} dr_{\alpha\beta} \times \exp(-\beta N \Phi) \right] , \tag{7}$$

with

$$\Phi = \frac{1}{2} \sum_{\alpha=1}^{4n} (m_\alpha)^2 + \frac{\alpha N - 1}{2\beta N} \mathrm{Tr} \ln \Lambda + \frac{i}{\beta N} \sum_{\alpha<\beta} r_{\alpha\beta} q_{\alpha\beta} - \frac{1}{\beta} \ln \left\langle\!\!\left\langle \sum_{[\theta^a]} \exp\left\{ \beta \sum_{a=1}^{n} \Big[A_a \cos(\theta^a - \xi) + \widetilde{A}_a \cos(\theta^a + \xi) + B_a \sin(\theta^a - \xi) + \widetilde{B}_a \sin(\theta^a + \xi) \Big] + i \sum_{\alpha<\beta} r_{\alpha\beta} B_{\alpha\beta} + 2\beta\omega \sum_a \theta^a \right\} \right\rangle\!\!\right\rangle_{\omega,\xi} ,$$

where $q_{\alpha\beta}$'s and $r_{\alpha\beta}$'s have been introduced for the integral representation of the δ functions, the matrix element of Λ is given by $\Lambda_{\alpha\beta} = (1-\beta/2)\delta_{\alpha\beta} - \beta q_{\alpha\beta}$, and $\langle\!\langle\,\rangle\!\rangle_{\omega,\xi}$ denotes the average over the quenched variables ω and ξ. Further, $B_{\alpha\beta}$'s have been defined so as to satisfy the relation

$$X_\mu = \exp\left[-\frac{\beta N}{2}\sum_{\alpha,\beta}(\delta_{\alpha\beta} - \beta B_{\alpha\beta}) m_\alpha^\mu m_\beta^\mu\right], \tag{8}$$

where

$$\mathbf{m}^\mu \equiv \left(A_1^\mu, A_2^\mu, \ldots, A_n^\mu, \widetilde{A}_1^\mu, \widetilde{A}_2^\mu, \ldots, \widetilde{A}_n^\mu, B_1^\mu, B_2^\mu, \ldots, B_n^\mu, \widetilde{B}_1^\mu, \widetilde{B}_2^\mu, \ldots, \widetilde{B}_n^\mu\right)$$

in the vector notation.

The mean-field equations follow from the saddle-point conditions and take the forms

$$\begin{aligned}
A_a &= \langle\!\langle \langle \cos(\theta^a - \xi)\rangle \rangle\!\rangle_{\omega,\xi}, \\
\widetilde{A}_a &= \langle\!\langle \langle \cos(\theta^a + \xi)\rangle \rangle\!\rangle_{\omega,\xi}, \\
B_a &= \langle\!\langle \langle \sin(\theta^a - \xi)\rangle \rangle\!\rangle_{\omega,\xi}, \\
\widetilde{B}_a &= \langle\!\langle \langle \sin(\theta^a + \xi)\rangle \rangle\!\rangle_{\omega,\xi}, \\
q_{\alpha\beta} &= \langle\!\langle N\langle B_{\alpha\beta}\rangle \rangle\!\rangle_{\omega,\xi}, \\
\bar{r}_{\alpha\beta} &= \frac{1}{\alpha}\left\langle\!\!\left\langle \sum_{\mu>1} m_\alpha^\mu m_\beta^\mu \right\rangle\!\!\right\rangle_{\omega,\xi},
\end{aligned} \tag{9}$$

where $\bar{r}_{\alpha\beta} \equiv i r_{\alpha\beta}/\alpha N\beta^2$ and $\langle O(\theta)\rangle$ stands for the average with respect to the action $\mathcal{L}([\theta])$,

$$\langle O(\theta)\rangle = \frac{\sum_{[\theta]} O(\theta) e^{\mathcal{L}([\theta])}}{\sum_{[\theta]} e^{\mathcal{L}([\theta])}},$$

with the action

$$\begin{aligned}
\mathcal{L}([\theta^a]) \equiv \beta\sum_a \Big[&A_a\cos(\theta^a - \xi) + \widetilde{A}_a\cos(\theta^a + \xi) \\
&+ B_a\sin(\theta^a - \xi) + \widetilde{B}_a\sin(\theta^a + \xi)\Big] \\
&+ i\sum_{\alpha<\beta} r_{\alpha\beta}B_{\alpha\beta} + 2\beta\omega\sum_a \theta^a.
\end{aligned}$$

The order parameters A_a and B_a measure the correlation between the state of the system and the pattern ($\mu = 1$), while $\widetilde{A}_a$ and $\widetilde{B}_a$ are irrelevant correlations and set equal to zero. Recalling the 2π periodicity of ξ, we set $A_a = B_a = m$ under the replica-symmetric ansatz together with $q_{\alpha\beta} = q$ and $\bar{r}_{\alpha\beta} = r$.

The free energy per oscillator is then expressed in terms of the replica-symmetric order parameters

$$\begin{aligned}
f = &\frac{1}{2}m^2 - \frac{\alpha q}{1-\beta(1-q)/2} + \frac{2\alpha}{\beta}\ln\left[1 - \frac{\beta}{2}(1-q)\right] \\
&+ \frac{\alpha\beta r}{2}(1-q) \\
&- \frac{1}{\beta}\int Dz_1\, Dz_2\, Dz_3\, \langle\!\langle \ln \mathrm{Tr}_{[\theta]} \exp(K\cos\theta \\
&+ L\sin\theta + M\theta)\rangle\!\rangle_\xi,
\end{aligned} \tag{10}$$

where K, L, and M have been introduced for simplicity

$$\begin{aligned}
K &\equiv \beta(m\cos\xi + \sqrt{\alpha r}\, z_1), \\
L &\equiv \beta(m\sin\xi + \sqrt{\alpha r}\, z_2), \\
M &\equiv 2\beta\sqrt{\sigma}\, z_3
\end{aligned}$$

and $\int Dz$ denotes the average over the normalized Gaussian variable z. The trace over θ can be easily performed with the help of the Fourier transformation in the compact interval $[-\pi, \pi]$ since the constant term is integrated to zero upon the Gaussian average over z_3. This leads to the saddle-point equations for order parameters in the form

$$\begin{aligned}
m &= \int Dz_1\, Dz_2\, Dz_3 \left\langle\!\!\left\langle \frac{I'(x)[K\cos\xi + L\sin\xi]}{xI(x)} \right\rangle\!\!\right\rangle_\xi, \\
1 - q &= \int Dz_1\, Dz_2\, Dz_3 \\
&\quad\times \left\langle\!\!\left\langle \frac{I'(x)}{xI(x)} + \frac{I''(x)}{I(x)} - \left[\frac{I'(x)}{I(x)}\right]^2 \right\rangle\!\!\right\rangle_\xi, \\
r &= \frac{q}{[1-\beta(1-q)/2]^2},
\end{aligned} \tag{11}$$

where we have defined

$$I(x) \equiv \sum_{n=-\infty}^{\infty} \frac{(-1)^n M^2}{M^2 + n^2} I_n(x),$$

with the modified Bessel function of the first kind I_n, $x \equiv \sqrt{K^2 + L^2}$, and $I'(x) \equiv \partial I(x)/\partial x$, etc. The order parameter m plays the role of the mean overlap while q and r correspond to the Edward-Anderson order parameter and the mean square random overlap, respectively.

III. PHASE BOUNDARIES

We now examine the numerical solutions of Eqs. (11) for several simple cases. We first consider the zero storage limit ($\alpha = 0$). In this limit, Eqs. (11) reduce to the simple form

$$m = \int Dz_3 \frac{I'(m\beta)}{I(m\beta)} \equiv f(m\beta), \tag{12}$$

which allows a nontrivial solution for $T < f'(0)$. Since the nontrivial solution $m \neq 0$ implies the appearance of correlation or synchronization in the system, the phase boundary separating the synchronized and the desynchronized phases on the (σ, T) plane is described by $T_c = f'(0)$, which in turn leads to

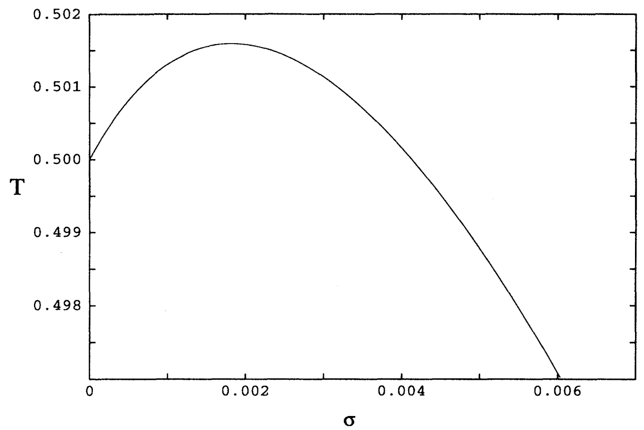

FIG. 1. Detailed phase boundary on the (σ, T) plane, displaying reentrance for small σ.

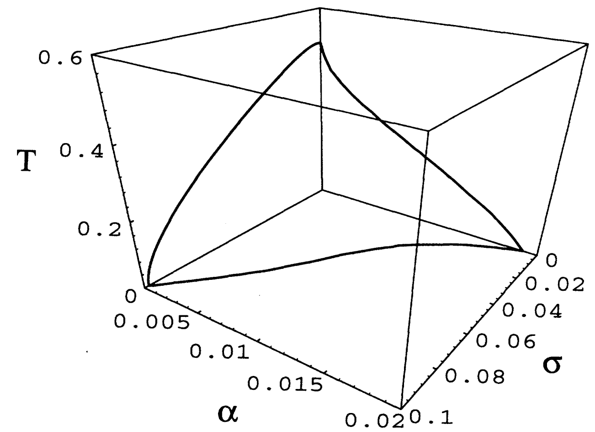

FIG. 2. Phase diagram in the (σ, α, T) space.

$$\begin{aligned}2x\sqrt{\sigma} &= \frac{3}{2} - \sqrt{\frac{\pi}{2}}xe^{2x^2}\mathrm{erfc}(\sqrt{2}x) \\ &\quad -\frac{3+x^2}{2}\left[1 - \sqrt{\frac{\pi}{2}}xe^{x^2/2}\mathrm{erfc}(x/\sqrt{2})\right] \\ &\equiv g(x),\end{aligned} \tag{13}$$

with $x \equiv T_c/2\sqrt{\sigma}$. For the given value of σ, the solution of Eq. (13) can be found numerically, giving the transition temperature T_c as a function of σ. Alternatively, at a given temperature the system may be considered to possess the critical value of σ, beyond which no synchronization appears. At zero temperature, the critical value of σ is determined by the equation $2\sqrt{\sigma_c} = g'(0) = \sqrt{\pi/8}$, which yields $\sigma_c = \pi/32$. The resulting phase boundary exhibits interesting reentrance phenomena in the small σ region, the detail of which is shown in Fig. 1. Namely, there exists a temperature range in which the system undergoes double transitions as σ is decreased: from the desynchronized phase to the synchronized one and to the desynchronized one again. Such reentrance implies that small nonuniformity in the natural frequencies helps the network to resist against external noises. It is expected that the reentrance becomes prominent as the storage α is increased, although this cannot be shown explicitly due to the difficulty in the numerical work for finite α. The phase boundary on the whole (σ, T) plane is shown on the $\alpha = 0$ plane of Fig. 2.

We next consider the case that all the neuronal oscillators are identical ($\sigma = 0$), which falls in with the Q-state clock model in the limit $Q \to \infty$. The corresponding phase boundary on the (α, T) plane, which is obtained from Eq. (11), indeed coincides with Fig. 5 of Ref. [8], except for the difference in the scale of α by a factor 2. (See the $\sigma = 0$ plane of the phase diagram shown in Fig. 2.) This difference originates from the fact that the Q-state clock model considers only the terms $\theta_i - \xi_i^\mu$, which is in contrast with our model including both the terms $\theta_i + \xi_i^\mu$ and $\theta_i - \xi_i^\mu$. The critical capacity $\alpha_c = 0.0189$ at $T = 0$ obtained here thus agrees well with the result $\alpha_c = 0.038$ of Ref. [8].

The zero temperature limit can also be investigated for arbitrary values of α and σ. At zero temperature, the direct expansion of Eqs. (11) fails because the arguments of the modified Bessel functions are divergent in all orders. Instead, we adopt the spin-wave approximation, which should be valid at zero temperature. Naively, we may expand $\cos\theta$ up to second order in θ, which yields, at $\alpha = 0$, the critical value $\sigma_c \simeq 0.074$; this is significantly smaller than the exact value obtained previously. We thus take into account the change of minimum due to the linear term and consider the expansion about the minimum of the washboard-type potential [9]

$$\begin{aligned}\exp(\sqrt{K^2+L^2}\cos\theta + M\theta) &\approx \exp\left[\widetilde{K}\cos(\theta-\theta_0)\right] \\ &\approx \exp\widetilde{K}\left[1 - \frac{1}{2}(\theta-\theta_0)^2\right],\end{aligned}$$

where $\theta_0 \equiv \sin^{-1}(M/\sqrt{K^2+L^2})$ and $\widetilde{K} \equiv \sqrt{K^2+L^2}\cos\theta_0$. This approximation gives the critical value $\sigma_c \simeq 0.098\,162$ at $\alpha = 0$, which shows excellent agreement with the exact value $\pi/32$. We thus use this accurate approximation to obtain the phase boundary on the (σ, α) plane. Figure 2 shows the overall phase diagram in the three-dimensional (σ, α, T) space, indicating the surface separating the synchronized and unsynchronized phases.

Finally, we consider the possibility of the glass transition, which is, for $\sigma = 0$, expected to occur above the synchronization transition line. We thus expand Eqs. (11) in powers of q and r, setting $m = \sigma = 0$. The glass transition temperature T_g is determined, to leading order, by the equation

$$q \simeq \frac{7}{8}\beta^2\alpha r \simeq \frac{7\beta^2\alpha q}{8(1-\beta/2)^2} + O(q^2), \tag{14}$$

which yields

$$T_g = \frac{1}{2} + \frac{\sqrt{14\alpha}}{4}. \tag{15}$$

It is of interest to note that this expansion fails for finite

σ, which indicates that the glass transition is of first order. The region of the glassy phase presumably grows as σ is increased, although the quantitative investigation is beyond the scope of this paper.

IV. CONCLUSION

We have studied the synchronization in a network of neuronal oscillators in the finite storage capacity regime. The system can be viewed as an attractor neural network that stores information via mutual phase locking. When all the oscillators are identical, the network shows behaviors qualitatively the same as those of the Q-state neural networks and the Hopfield model. Phase locking has been shown to be destroyed if the storage exceeds the critical value that depends on the temperature. The transition line separating the synchronized and the desynchronized phases has been calculated numerically and the continuous transition above the line to the glass phase has been pointed out. Here the distribution of natural frequencies makes the glass transition discontinuous.

In particular, the appearance of the reentrant behavior in the zero storage limit reflects the subtle interplay between the two kinds of fluctuations present in the system. In usual disordered systems such as the Sherrington-Kirkpatrick spin-glass model [11] and the Hopfield model [12], the reentrance is in general believed to be characteristic of the replica-symmetric calculation. Such reentrant behavior has been found at extremely low temperatures, where the replica-symmetric solution becomes unphysical and consideration of replica-symmetry breaking is required; the true replica-symmetry-broken solution removes the reentrance and increases the capacity of the system. In the system considered here, on the other hand, the reentrance appears at relatively high temperatures even in the zero storage limit, which raises the possibility that the reentrance here may have a different origin. For clarification, a detailed investigation of the stability of the solution and the glass transition should be performed, together with extensive numerical calculations in the full (σ, α, T) space.

ACKNOWLEDGMENTS

This work was supported in part by the SNU-Daewoo Research Fund, by the Basic Science Research Institute Program, Ministry of Education of Korea, and by the Korea Science and Engineering Foundation through the SRC program.

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